

L^p –solutions of the stochastic transport equation

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Abstract

We consider the stochastic transport linear equation and we prove existence and uniqueness of weak L^p –solutions. Moreover, we obtain a representation of the general solution and a Wong-Zakai principle for this equation. We make only minimal assumptions, similar to the deterministic problem. The proof is supported on the generalized Itô-Ventzel-Kunita formula (see [15]) and the theory of Lions-DiPerna on transport linear equation (see [9]).

Key words: Stochastic perturbation, Transport equation, Itô formula.

MSC2000 subject classification: 60H10 , 60H15 .

1 Introduction

In this article we establish global existence and uniqueness of solution of the transport linear equation with a stochastic perturbation. Namely, we consider the following equation:

$$\begin{cases} \frac{d}{dt}u(t, x) + b(t, x)\nabla u(t, x) + \nabla u(t, x)\frac{dB_t}{dt} = 0, \\ u(0, x) = u_0(x) \in L^p(\mathbb{R}^d), \end{cases} \quad (1)$$

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where $B_t = (B_t^1, \dots, B_t^d)$ is a standard Brownian motion in \mathbb{R}^d and the stochastic integration is taken in the Stratonovich sense.

This equation has been treated for the case $u_0(x) \in L^\infty(\mathbb{R}^d)$ (see [12] and [16]) via the stochastic characteristic method. Our aim here is to prove the existence, uniqueness and regularity when the initial data $u_0(x) \in L^p(\mathbb{R}^d)$ for $p \in [1, \infty)$. Some partial results are presented in [17], where the case $u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ was studied.

The theory of renormalized solutions of the linear transport equation was introduced by DiPerna and Lions in a celebrated paper [9]. They deduced the existence, uniqueness and stability results for ordinary differential equations with rough coefficients from corresponding results on the associated linear transport equation. Similar results were obtained in [7] by taking the standard Gaussian measure as the reference measure. Ambrosio [2] generalized the results to the case where the coefficients have only bounded variation regularity by considering the continuity equation. These results have recently been generalized into different settings, [3] and [10] for infinite dimensional spaces, [11] and [20] for generalizations to transport-diffusion equations and its associated stochastic differential equations.

We prove existence and uniqueness of weak L^p -solution using the generalized Itô-Ventzel-Kunita formula (see Theorem 8.3 of [15]) and the results on existence and uniqueness for the deterministic transport linear equation (see for example [9] and [17]). We give a Wong-Zakai principle for the stochastic transport equation (1), this principle is proved via stability properties of the deterministic transport linear equation. We would like to mention that our approach clearly differs from that one in [12], however this article has been a source of inspiration for us.

The plan of exposition is as follows: In section 2 we prove existence of weak L^p -solutions and we point some extensions. In section 3, we show a uniqueness theorem for weak L^p -solutions. Finally, in section 4, we establish

a Wong-Zakai principle for the SPDE (1).

Through of this paper we fix a stochastic basis with a d -dimensional Brownian motion $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \in [0, T]\}, \mathbb{P}, (B_t))$.

2 Stochastic transport equation. Existence of weak solutions

Definition 2.1 *A weak L^p -solution of the Cauchy problem (1) is a stochastic process $u \in L^\infty(\Omega \times [0, T], L^p(\mathbb{R}^d))$ such that, for every test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, the process $\int u(t, x)\varphi(x)dx$ has a continuous modification which is a \mathcal{F}_t -semimartingale and satisfies*

$$\begin{aligned} \int u(t, x)\varphi(x)dx &= \int u_0(x)\varphi(x) dx \\ &+ \int_0^t \int b(s, x)\nabla\varphi(x)u(s, x) dx ds + \int_0^t \int \operatorname{div} b(s, x)\varphi(x)u(s, x) dx ds \\ &+ \sum_{i=1}^d \int_0^t \int D_i\varphi(x)u(s, x) dx \circ dB_s^i \end{aligned}$$

We shall always assume that

$$b \in L^1([0, T], (L_{loc}^1(\mathbb{R}^d))^d) \quad (2)$$

We observe that this definition makes sense if we assume

$$b \in L^1([0, T], (L_{loc}^q(\mathbb{R}^d))^d) \quad (3)$$

where q is the conjugate exponent of p .

Lemma 2.1 *Let $p \in [1, \infty)$, $u_0 \in L^p(\mathbb{R}^d)$. Assume (2), (3) and that*

$$\operatorname{div} b \in L^1([0, T], L^\infty(\mathbb{R}^d)) \quad (4)$$

Then there exists a weak L^p -solution u of the SPDE (1).

Proof: Step 1 (auxiliary transport equation) We consider the following auxiliary transport equation

$$\begin{cases} v_t + b(t, x + B_t) \nabla v(t, x) &= 0 \\ v(0, x) &= u_0(x), \quad x \in \mathbb{R}^d. \end{cases} \quad (5)$$

According to an easy modification of [9], Proposition II.1 (taking only test functions defined on \mathbb{R}^d) there is a solution $v \in L^\infty([0, T] \times \Omega, L^p(\mathbb{R}^d))$ of the equation (5) in the sense that it satisfies

$$\begin{aligned} & \int v(t, x) \varphi(x) dx = \int u_0(x) \varphi(x) dx \\ & + \int_0^t \int b(s, x + B_s) \nabla \varphi(x) v(s, x) dx ds + \int_0^t \int \operatorname{div} b(s, x + B_s) \varphi(x) v(s, x) dx ds \end{aligned} \quad (6)$$

Step 2 (Solution via Itô-Ventzel-Kunita formula)

Applying the Itô-Ventzel-Kunita formula to $F(y) = \int u(t, x) \varphi(x + y) dx$ (see Theorem 8.3 of [15]) we obtain that

$$\int v(t, x) \varphi(x + B_t) dx$$

is equal to

$$\begin{aligned} & \int u_0(x) \varphi(x) dx + \int_0^t \int b(s, x + B_s) \nabla \varphi(x + B_s) v(s, x) dx ds \\ & + \int_0^t \int \operatorname{div} b(s, x + B_s) \varphi(x + B_s) v(s, x) dx ds \\ & + \sum_{i=1}^d \int_0^t \int v(s, x) \frac{\partial}{\partial y_i} \varphi(x + B_s) dx \circ dB_s^i. \end{aligned}$$

We note that $\frac{\partial}{\partial y_i} \varphi(x + B_s) = \frac{\partial}{\partial x_i} \varphi(x + B_s)$. Thus

$$\begin{aligned}
& \int v(t, x) \varphi(x + B_t) dx = \int u_0(x) \varphi(x) dx \\
& + \int_0^t \int b(s, x + B_s) \nabla \varphi(x + B_s) v(s, x) dx ds + \int_0^t \int \operatorname{div} b(s, x + B_s) \varphi(x + B_s) v(s, x) dx ds \\
& + \sum_{i=1}^d \int_0^t \int v(s, x) D_i \varphi(x + B_s) dx \circ dB_s^i. \tag{7}
\end{aligned}$$

From the equation (7) we follow that $u(t, x) := v(t, x - B_t)$ is a weak L^p -solution of the SPDE (1).

Remark 2.1 *We observe that the same proof is valid if we assume that $b(t, x, \cdot) \in \mathcal{F}_t$ for all $(t, x) \in [0, T] \times \mathbb{R}^d$ and verifies (3) and (4) almost surely for $\omega \in \Omega$. This results gives a partial answer about the existence of solution for the SPDE (1) with stochastic coefficient (see Introduction of [12])*

Remark 2.2 *We would to note that the same proof works for the equation*

$$\begin{cases} \frac{d}{dt} u(t, x) + b(t, x) \nabla u(t, x) + \nabla u(t, x) \frac{dB_t}{dt} + c(t, x) u = f(t, x), \\ u(0, x) = u_0(x) \in L^p(\mathbb{R}^d), \end{cases} \tag{8}$$

where c and f satisfy the conditions of the Proposition II.1 of [9]

Remark 2.3 *We mention some future works*

- a) *The case that $b(t, \cdot, \omega)$ is a nonadapted process could be studied via a Generalized Itô formula for nonadapted process (see by example [18]).*
- b) *The stochastic transport equations with other noises could be studied via the stochastic calculus via regularization (see [8] and [13]).*
- c) *For initial data and coefficients more singular, a possible approach is to study the transport equation in the sense of generalized function algebras, see for instance [1] and [19]. For a new approach see [5] and [6].*

3 Uniqueness

In this section, we shall present a uniqueness theorem for the SPDE (1) under similar conditions to the deterministic case (see for instance [9] and [17]).

Theorem 3.1 *Let $p \in [1, \infty)$. Assume that $\operatorname{div} b \in L^1([0, T], L^\infty(\mathbb{R}^d))$, $b \in L^1([0, T], (W_{loc}^{1,q}(\mathbb{R}^d))^d)$ and $\frac{|b|}{1+|x|} \in L^1([0, T], L^1(\mathbb{R}^d)) + L^1([0, T], L^\infty(\mathbb{R}^d))$. Then, for every $u_0 \in L^p(\mathbb{R}^d)$ there exists a unique weak L^p -solution of the Cauchy problem (1).*

Proof: By linearity we have to show that a weak L^p -solution with initial condition $u_0(x) = 0$ vanishes identically. Applying the Itô-Ventzel-Kunita formula (see Theorem 8.3 of [15]) to $F(y) = \int u(t, x) \varphi(x - y) dx$, we obtain that

$$\int u(t, x) \varphi(x - B_t) dx$$

is equal to

$$\begin{aligned} & \int_0^t \int b(s, x) \nabla \varphi(x - B_s) u(s, x) dx ds + \int_0^t \int \operatorname{div} b(s, x) \varphi(x - B_s) u(s, x) dx ds \\ & + \sum_{i=1}^d \int_0^t \int u(s, x) D_i \varphi(x - B_s) dx \odot dB_s^i + \sum_{i=1}^d \int_0^t \int u(s, x) \frac{\partial}{\partial y_i} [\varphi(x - B_s)] dx \odot dB_s^i. \end{aligned}$$

We observe that $\frac{\partial}{\partial y_i} [\varphi(x - B_s)] = -\frac{\partial}{\partial x_i} \varphi(x - B_s)$. Thus $V(t, x) = u(t, x + B_t)$ verifies

$$\begin{aligned} \int V(t, x) \varphi(x) dx &= \int_0^t \int b(s, x + B_s) \nabla \varphi(x) V(s, x) dx ds \\ &+ \int_0^t \int \operatorname{div} b(s, x + B_s) \varphi(x) V(s, x) dx ds. \end{aligned}$$

Let ϕ_ε be a standard mollifier. Since $b(s, x + B_s)$ satisfies \mathbb{P} a.s the hypothesis of our Theorem, then by the Commuting Lemma (see Lemma II.1 of [9]), $V_\varepsilon(t, x) = V(t, \cdot) * \phi_\varepsilon$ verifies

$$\lim_{\varepsilon \rightarrow 0} \frac{dV_\varepsilon}{dt} + b(t, x + B_t) \nabla V_\varepsilon = 0 \quad \mathbb{P} \text{ a.s in } L^1([0, T], L^1_{loc}(\mathbb{R}^d)).$$

We deduce that if $\beta \in C^1(\mathbb{R})$ and β' is bounded, then

$$\frac{d\beta(V)}{dt} + b(t, x + B_t) \nabla \beta(V) = 0. \quad (9)$$

Now, following the same steps in the proof of Theorem II. 2 of [9], we define for each $M \in (0, \infty)$ the function $\beta_M(t) = (|t| \wedge M)^p$ and obtain that

$$\frac{d}{dt} \int \beta_M(V(t, x)) dx \leq C \int \beta_M(V(t, x)) dx.$$

Taking expectation we have that

$$\frac{d}{dt} \int \mathbb{E}(\beta_M(V(t, x))) dx \leq C \int \mathbb{E}(\beta_M(V(t, x))) dx.$$

From Gronwall Lemma we conclude that $\beta_M(V(t, x)) = 0$. Thus $u = 0$.

Remark 3.1 *We observe that the unique solution $u(t, x)$ has the representation $u(t, x) = v(t, x - B(t))$, where v satisfies (6). From Corollary 2.2 of [9], we follow that v belongs to $C([0, T], L^p(\mathbb{R}^d))$. Thus $u \in C([0, T], L^p(\mathbb{R}^d))$.*

4 Wong-Zakai principle

The Wong-Zakai principle says that the solutions to equations where the noise is approximated by more regular processes converge to the solution of the stochastic differential equation with Stratonovich integrals. We mention that there exist several works about of the Wong-Zakai principle for SPDE (see for instance [4], [14] and references). Our method for prove this principle is based

on the stability properties for the renormalized solutions of the deterministic transport equation.

Now, we consider approximations of the Brownian motion, by continuous and bounded variation processes $B_n(t)$ such that

$$\lim_{n \rightarrow \infty} B_n(t) = B(t) \quad \mathbb{P} - a.s. \quad \text{uniformly in } t.$$

We consider the next equations

$$\begin{cases} \frac{d}{dt} u_n(t, x) + b(t, x) \nabla u_n(t, x) + \nabla u_n(t, x) \frac{dB_n}{dt}(t) = 0, \\ u_n(0, x) = u_0(x) \in L^p(\mathbb{R}^d). \end{cases} \quad (10)$$

Definition 4.1 *A weak L^p -solution of the Cauchy problem (10) is a stochastic process $u_n \in L^\infty(\Omega \times [0, T], L^p(\mathbb{R}^d))$ such that, for every test function $\varphi \in C_0^\infty(\mathbb{R}^d)$, the process $\int u_n(t, x) \varphi(x) dx$ has a continuous modification and satisfies*

$$\begin{aligned} \int u_n(t, x) \varphi(x) dx &= \int u_0(x) \varphi(x) dx \\ &+ \int_0^t \int b(s, x) \nabla \varphi(x) u_n(s, x) dx ds + \int_0^t \int \operatorname{div} b(s, x) \varphi(x) u_n(s, x) dx ds \\ &+ \sum_{i=1}^d \int_0^t \int D_i \varphi(x) u_n(s, x) dx dB_n^i(s). \end{aligned}$$

Lemma 4.1 *Let $u_0 \in L^p(\mathbb{R}^d)$ where $p \in [1, \infty)$. Suppose that $\operatorname{div} b \in L^1([0, T], L^\infty(\mathbb{R}^d))$ and (2) and (3) holds. Then there exists a weak L^p -solution u_n of the SPDE (10).*

Proof: We consider the following auxiliary transport equation

$$\begin{cases} \frac{dv_n}{dt} &= b(t, x + B_n(t)) \nabla v_n(t, x) \\ v_n(0, x) &= u_0(x). \end{cases} \quad (11)$$

According to a small modification of [9], Proposition II.1 there is a solution $v_n(t, x)$ of the equation (11) in the sense that

$$\int v_n(t, x) \varphi(x) dx$$

is equal to

$$\begin{aligned} & \int u_0(x) \varphi(x) dx + \int_0^t \int b(s, x + B_n(s)) \nabla \varphi(x) v_n(s, x) dx ds \\ & + \int_0^t \int \operatorname{div} b(s, x + B_n(s)) \varphi(x) v_n(s, x) dx ds \end{aligned} \quad (12)$$

Following the proof of the Lemma 4.1 we get that

$$u_n(t, x) = v_n(t, x - B_n(t)) \quad (13)$$

is a weak $L^p(\mathbb{R}^d)$ -solution of the SPDE (10).

The uniqueness of the approximate problem (10), follows changing B by B_n in the proof of the Theorem 3.1.

Theorem 4.1 *Let $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\operatorname{div} b \in L^1([0, T], L^\infty(\mathbb{R}^d))$, $b \in L^1([0, T], (W_{loc}^{1,q}(\mathbb{R}^d))^d)$ and $\frac{|b|}{1+|x|} \in L^1([0, T], L^1(\mathbb{R}^d)) + L^1([0, T], L^\infty(\mathbb{R}^d))$. Then, for every $u_0 \in L^p(\mathbb{R}^d)$ there exists a unique weak L^p -solution of the Cauchy problem (10)*

Finally, we prove our Wong-Zakai principle.

Theorem 4.2 *Let $p \in [1, \infty)$ and $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\operatorname{div} b \in L^1([0, T], L^\infty(\mathbb{R}^d))$, $b \in L^1([0, T], (W_{loc}^{1,q}(\mathbb{R}^d))^d)$ and $\frac{|b|}{1+|x|} \in L^1([0, T], L^1(\mathbb{R}^d)) + L^1([0, T], L^\infty(\mathbb{R}^d))$. Let u and u_n are weak L^p -solutions of the SPDE (1) and (10) respectively. Then u_n converges to u , \mathbb{P} a.s in $C([0, T], L^p(\mathbb{R}^d))$.*

Proof: We know that $u(t, x) = v(t, x - B(t))$ and $u_n(t, x) = v_n(t, x - B_n(t))$ where $v(t, x)$ and $v_n(t, x)$ satisfies (6) and (12) respectively. By Theorem 2.4 of [9] we have that

$$\lim_{n \rightarrow \infty} v_n(t, x) = v(t, x) \quad \mathbb{P} \text{ a.s in } C([0, T], L^p(\mathbb{R}^d)).$$

From this fact we obtain immediately that

$$\lim_{n \rightarrow \infty} u_n(t, x) = u(t, x) \quad \mathbb{P} \text{ a.s in } C([0, T], L^p(\mathbb{R}^d)).$$

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